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Exact spinon excitations in an integrable ferrimagnetic spin chain

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Abstract
An exactly solvable one-dimensional ferrimagnetic quantum spin chain model with alternating 1/2-spins and 1-spins is proposed and solved via algebraic Bethe ansatz. It is found that the ground state of this model is a ferrimagnetic state with a residual magnetization. It is proven that the low-lying elementary excitations of the system are the varieties of gapless spinons rather than spin waves.

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1. Introduction

The one-dimensional (1D) quantum models have many applications in the fields of quantum magnetism, strongly correlated electron systems and trapped cold atoms. Besides some quasi-1D materials, more and more artificial 1D systems have been realized in experiments by means of the modern micro-fabrication techniques or magnetic and laser traps. The 1D quantum systems usually exhibit unique properties compared to their counterparts in high dimensions. For instance, for the gapless 1D systems, the low-energy physics can be described by the so-called Luttinger liquid [1] and the elementary excitations usually carry fractional charges. For the quantum spin chains with integer spins, Haldane conjectured that there is a finite gap in the energy spectrum [2]. Interestingly, some 1D quantum models can be solved exactly, which provided us with a deep understanding of the 1D quantum systems. Typical integrable quantum spin chains are the spin-1/2 Heisenberg model [3], $SU(N)$-invariant high-spin chain [4–6] and the $SU(2)$-invariant high-spin chains [7–10].

Besides the homogeneous spin chains, there also exist some alternating spin systems. In experiments, the 1D alternating spin systems are realized in some bimetallic chain compounds such as $ACu(pba)(H_2O)_3\cdot nH_2O$ and $ACu(pbaOH)(H_2O)_3\cdot nH_2O$, where A=Mn, Fe, Co, Ni, Zn, pba=1, 3-propylenebis (oxamato), and pbaOH=2-hydroxy-1, 3-propylenebis (oxamato)

[11, 12]. Motivated by the progress in experiments, the theoretical studies of ferrimagnetic spin chains are carried out extensively. For instance, Kolezhuk *et al* studied the ground state of the Heisenberg spin chain with alternating spin-1 and spin-1/2 and antiferromagnetic exchange interactions between nearest neighbors by constructing the matrix-product states [13]. Wu *et al* studied the ground-state properties of the quantum ferrimagnetic spin chain and found that the ground state of the system has the ferrimagnetic long-range order [14]. The low-temperature properties of the alternating spin chain with antiferromagnetic nearest neighbor exchange couplings are studied by Brehmer *et al* [15]. Pati *et al* investigated the low-lying excited states with a renormalization group method [16, 17]. Yamamoto *et al* study the elementary excitations, specific heat and magnetic properties of the alternating spin system with nearest neighbor interactions by using the density-matrix renormalization group technique as well as the quantum Monte Carlo simulation [18–20]. It is also found that the specific heat of low-dimensional quantum ferrimagnets has the double-peak structure [21]. Recently, in an interesting work, Mohakud *et al* study the ground-state properties and low-lying excitations of an alternating spin-1 and spin-1/2 model with nearest antiferromagnetic coupling and next nearest neighbor ferromagnetic coupling via the spin wave theory, density matrix renormalization group and numerical exact diagonalization methods [22].

Unfortunately, the ferrimagnetic spin chain with nearest neighbor interactions cannot be solved exactly. However, by adding some additive terms such as the multi-spin interactions, de Vega and Woynarovich constructed an integrable model with alternating 1/2 and 1 spins [23]. By using the Bethe ansatz method, they demonstrated that the ground state of their model is a spin singlet [24]. Fujii *et al* studied the magnetic and critical properties of this integrable system in magnetic fields [25]. Doikou studied the de Vega–Woynarovich model in the presence of a quantum impurity and solved the system with both diagonal and non-diagonal integrable open boundary conditions [26]. The anisotropic case of the de Vega–Woynarovich model is studied by Dörfel and Meißner, who obtained the ground state, the low excitations and the conformal invariance of the system [27, 28]. Bytsko and Doikou generalized the de Vega–Woynarovich model and studied the ground state, thermodynamics and conformal properties of anisotropic XXZ spin chains with alternating spins 1/2 and S via the algebraic Bethe ansatz method [29]. Very recently, Ribeiro and Klümper constructed a general integrable quantum spin chains with alternating spins S_1 and S_2 [30]. Another integrable spin-1 and spin-1/2 mixed model is proposed by Aladim and Martins [31]. Later, it is shown that the spin-1 operators can be generalized to the arbitrary case while the integrability is kept [32]. Other interesting topics on the integrable alternating spin systems can be found in some other works [33, 34]. In the exactly solvable ferrimagnetic quantum spin chains, the basic terms are the couplings between the spin-1/2 and the spin-1. The integrability of the systems requires that the Hamiltonian must have some attached terms. The attached terms of the above-mentioned integrable ferrimagnetic systems are different due to the different constructions.

In this paper, we propose an integrable ferrimagnetic spin chain model with alternating 1/2 and 1 spins. The model is solved via the algebraic Bethe ansatz [35]. Based on the exact solutions, we found that the ground state is a ferrimagnetic state with a finite residual magnetization and the low-lying elementary excitations of the system are spinons as those in the spin-1/2 Heisenberg spin chain.

The paper is organized as follows. In section 2, we introduce the model and derive the exact solutions. In section 3, we discuss the ground-state properties of the system. In section 4, we study the low-lying excitations. Section 5 is attributed to the concluding remarks.

2. The model

We consider a 1D ferrimagnetic quantum spin chain with the following model Hamiltonian:

$$H = \sum_{n=1}^{N/2} \Big[\boldsymbol{\sigma}_{2n-1} \cdot \mathbf{S}_{2n} + \mathbf{S}_{2n} \cdot \boldsymbol{\sigma}_{2n+1} - \frac{7}{8} \boldsymbol{\sigma}_{2n-1} \cdot \boldsymbol{\sigma}_{2n+1} + \frac{1}{2} (\boldsymbol{\sigma}_{2n-1} \cdot \mathbf{S}_{2n})(\mathbf{S}_{2n} \cdot \boldsymbol{\sigma}_{2n+1}) + \frac{1}{2} (\mathbf{S}_{2n} \cdot \boldsymbol{\sigma}_{2n+1})(\boldsymbol{\sigma}_{2n-1} \cdot \mathbf{S}_{2n}) \Big], \tag{1}$$

where $\boldsymbol{\sigma}_n$ and $\mathbf{S}_n$ are the Pauli matrices and the spin-1 operators at site n, respectively; the even number N is the length of the chain. Here the periodic boundary condition $\boldsymbol{\sigma}_{N+1} = \boldsymbol{\sigma}_1$ is assumed. The Hamiltonian contains three kinds of terms, i.e., the nearest neighbor antiferromagnetic coupling between the Pauli spins and the 1 spins, the next nearest neighbor ferromagnetic coupling between Pauli spins and a three-spin coupling term to ensure the integrability of the model. The four-spin couplings come from the spin–phonon interactions, and is irrelevant to the ground state of the system [36]. The four-spin interacting terms are relevant and may significantly affect the low-energy properties of the system. Obviously, the Hamiltonian (1) is $SU(2)$-invariant.

It is well known that the integrability of the 1D lattice models is related to the transfer matrix of the corresponding two-dimensional vertex model. In order to show the integrability of the present system (1) clearly, we define the Lax operators

$$L_{0n}^{\sigma} = \frac{1}{\lambda + \eta} (\lambda + \eta P_{0n}^{\sigma}), \tag{2}$$

$$L_{0n}^{S} = \frac{1}{\lambda + \frac{3}{2}\eta} \left(\lambda + \frac{3}{2} \eta P_{0n}^{S} \right), \tag{3}$$

where λ is the spectral parameter, η is the crossing parameter, $P_{0n}^{\sigma} = (1 + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_n)/2$ is the spin permutation operators, 0 describes the auxiliary space and n describes the quantum (site) space, $P_{0n}^{S} = (1 + 2\boldsymbol{\sigma}_0 \cdot \mathbf{S}_n)/3$ is an auxiliary operator. The L-operators (2) and (3) are quite different. The operator (2) acts on the space $V_0 \otimes V_n$, where V_0 is the auxiliary space of spin-1/2 which is a 2×2 matrix and V_n is the quantum space of spin-1/2 which is also a 2×2 matrix. If the spectrum parameter λ is zero, the L-operator (2) exactly degenerates into the permutation operator, which permutes two 1/2 spins. Thus the L-operator (2) can be written as a 4×4 matrix. The L-operator (3) is different from equation (2). The operator (3) expresses the interactions between the spin-1/2 and spin-1. The operator (3) acts on the space of $V_0 \otimes V_m$, where V_0 is the auxiliary space again which is a 2×2 matrix and V_m is the quantum space of spin-1 which is a 3×3 matrix. Thus the L-operator (3) can be written as a 6×6 matrix. If the spectrum parameter λ is zero, the L-operator (3) is not the permutation operator and it does not permute the 1/2 and 1 spins. The Lax operators (2) and (3) satisfy the Yang–Baxter relations

$$L_{00'}^{\sigma}(\lambda - \mu) L_{0n}^{\sigma}(\lambda) L_{0'n}^{\sigma}(\mu) = L_{0'n}^{\sigma}(\mu) L_{0n}^{\sigma}(\lambda) L_{00'}^{\sigma}(\lambda - \mu),$$
$$L_{00'}^{\sigma}(\lambda - \mu) L_{0n}^{S}(\lambda) L_{0'n}^{S}(\mu) = L_{0'n}^{S}(\mu) L_{0n}^{S}(\lambda) L_{00'}^{\sigma}(\lambda - \mu).$$

The monodromy matrix of the system is constructed as

$$T_0(\lambda) = L_{01}^{\sigma}(\lambda) L_{02}^{S}(\lambda) L_{03}^{\sigma}(\lambda) L_{04}^{S}(\lambda) \cdots L_{0N}^{S}(\lambda). \tag{4}$$

Note that the spin-1/2 operators and the spin-1 operators in the quantum space are arranged alternatively, and share the same auxiliary space expanded by the Pauli matrix σ_0. It is easy to check that the monodromy matrix (4) also satisfies the following Yang-Baxter relation

$$L_{00'}^{\sigma}(\lambda - \mu) T_0(\lambda) T_{0'}(\mu) = T_{0'}(\mu) T_0(\lambda) L_{00'}^{\sigma}(\lambda - \mu). \tag{5}$$

Define the transfer matrix $\tau(\lambda) = tr_0 T_0(\lambda)$. From the Yang–Baxter relation (5), we find that the transfer matrices with different spectral parameters commute with each other, $[\tau(\lambda), \tau(\mu)] = 0$. Thus the system (1) has an infinite number of conserved quantities and is integrable. The explicit expression of the Hamiltonian (1) is obtained by the derivative of logarithm of the transfer matrix as

$$H = \frac{9}{4}\eta \frac{\partial}{\partial\lambda} \ln \tau(\lambda)\bigg|_{\lambda=0} + \frac{17}{16}N. \tag{6}$$

We use the algebraic Bethe ansatz method to diagonalize the Hamiltonian (1). For simplicity, we write the matrix form of the monodromy matrix in the auxiliary space

$$T_0(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{7}$$

From the Yang–Baxter relation (5) we obtain the following commutation relations

$$A(\lambda)B(\mu) = \frac{\lambda - \mu - \eta}{\lambda - \mu} B(\mu)A(\lambda) + \frac{\eta}{\lambda - \mu} B(\lambda)A(\mu),$$
$$D(\lambda)B(\mu) = \frac{\lambda - \mu + \eta}{\lambda - \mu} B(\mu)D(\lambda) - \frac{\eta}{\lambda - \mu} B(\lambda)D(\mu),$$
$$B(\lambda)B(\mu) = B(\mu)B(\lambda).$$

Choosing the vacuum state of the system as the direct production of the local spin-up state (ferromagnetic state)

$$|0\rangle = |1/2\rangle_1 \otimes |1\rangle_2 \otimes |1/2\rangle_3 \otimes |1\rangle_4 \cdots \otimes |1\rangle_N, \tag{8}$$

the monodromy matrix acting on the vacuum state gives

$$A(\lambda)|0\rangle = |0\rangle, \tag{9}$$

$$D(\lambda)|0\rangle = \frac{\lambda^{\frac{N}{2}} \left(\lambda - \frac{1}{2}\eta\right)^{\frac{N}{2}}}{(\lambda + \eta)^{\frac{N}{2}} \left(\lambda + \frac{3}{2}\eta\right)^{\frac{N}{2}}} |0\rangle, \tag{10}$$

$$C(\lambda)|0\rangle = 0. \tag{11}$$

The element $B(\lambda)$ acting on the vacuum state gives nonzero value and can be regarded as the spin-flip operator. Assume that the eigenstates of the system take the following form

$$|\mu_1, \ldots, \mu_M\rangle = B(\mu_1) \cdots B(\mu_M)|0\rangle, \tag{12}$$

where M is the number of flipped spins. By acting the transfer matrix $\tau(\lambda) = A(\lambda) + D(\lambda)$ on the assumed states and using the commutation relations among the elements of the monodromy matrix, we obtain two kinds of terms. One gives the wanted terms and the other gives the unwanted terms. If the assumed states (12) are the eigenstates of the transfer matrix, the unwanted terms must be canceled , which gives the Bethe ansatz equations or the constraints of the states (12). From the eigenvalues of transfer matrix and the relation between the transfer matrix and the Hamiltonian, we obtain the eigenvalues of the system (1) as

$$E = \frac{47}{16}N - \frac{9}{4}\sum_{j=1}^{M} \frac{1}{\lambda_j^2 + \frac{1}{4}}, \tag{13}$$

where the parameters λ_j should satisfy the following Bethe ansatz equations

$$\left[\frac{(\lambda_j - \mathrm{i}/2)(\lambda_j - \mathrm{i})}{(\lambda_j + \mathrm{i}/2)(\lambda_j + i)}\right]^{\frac{N}{2}} = \prod_{l=1, l\neq j}^{M} \frac{\lambda_j - \lambda_l - \mathrm{i}}{\lambda_j - \lambda_l + i}, \tag{14}$$

where $j = 1, 2, \ldots, M$. The properties of the model (1) are uniquely determined by equations (15) and (16).

3. Ground state

By taking the logarithm of the Bethe ansatz equation (14), we readily arrive at

$$\theta_1(\lambda_j) + \theta_2(\lambda_j) = \frac{2}{N} \sum_{l=1, l\neq j}^{M} \theta_2(\lambda_j - \lambda_l) + \frac{4\pi I_j}{N}, \tag{15}$$

where $\theta_n(x) = 2\arctan(2x/n)$ and the quantum number I_j is an integer if M is odd and is half-odd if M is even. Define

$$Z(\lambda) = \frac{1}{2\pi}\left[\frac{\theta_1(\lambda)}{2} + \frac{\theta_2(\lambda)}{2} - \frac{1}{N}\sum_{l=1}^{M}\theta_2(\lambda - \lambda_l)\right], \tag{16}$$

then the Bethe ansatz equations (15) take the form of $Z(\lambda_j) = I_j/N$. In the thermodynamic limit, the number of total spins N and the number of flipped spins M tend to infinity while the ratio M/N keeps a non-zero value. Then the quantum number I_j takes the continue values. In the thermodynamic limit, we define the $\sigma(\lambda)$ as the density of number of flipped spins

$$\sigma(\lambda) = \frac{\mathrm{d}Z(\lambda)}{\mathrm{d}\lambda}. \tag{17}$$

The $\sigma(\lambda)$ describes the density of quasi-particles. In the ground state, the particles are filled below the Fermi surface thus all the $\{I_j\}$s are consecutive numbers around zero. The corresponding density of particles $\sigma(\lambda)$ should satisfy the integral equation,

$$\sigma(\lambda) = \frac{1}{2}a_1(\lambda) + \frac{1}{2}a_2(\lambda) - \int_{-\Lambda}^{\Lambda} a_2(\lambda - \lambda')\sigma(\lambda')\,\mathrm{d}\lambda', \tag{18}$$

where $a_n(\lambda) = n/[2\pi(\lambda^2 + n^2/4)]$ and Λ is the Fermi point. Because real λ contributes positive energy, in the ground state, Λ must tend to infinity. In this case, equation (18) can be solved through the Fourier transformation. After some algebra, we have

$$\widetilde{\sigma}(\omega) = \frac{1}{2}\frac{e^{-\frac{|\omega|}{2}} + e^{-|\omega|}}{1 + e^{-|\omega|}}, \tag{19}$$

and

$$\sigma(\lambda) = \frac{1}{4\cosh(\pi\lambda)} + \frac{1}{4\cosh\left(\pi\lambda - \frac{\pi}{2}\right)}. \tag{20}$$

It is easy to show that

$$\frac{M}{N} = \int \sigma(\lambda)\,\mathrm{d}\lambda = \frac{1}{2}. \tag{21}$$

The residual magnetization therefore reads $S^z = 3N/4 - M = N/4$, indicating a ferrimagnetic ground state. The corresponding energy density of this state is

$$\begin{aligned}\frac{E}{N} &= \frac{47}{16} - \frac{9}{2}\pi\int a_1(\lambda)\sigma(\lambda)\,\mathrm{d}\lambda \\ &= -\frac{25}{16} - \frac{9}{4}\ln 2 + \frac{9}{4}\pi.\end{aligned} \tag{22}$$

4. Elementary excitations

Now let us turn to the elementary excitations of the system. Generally, the basic excitation energy quanta in the integrable models are described by the so-called dressed energy which in our case reads

$$\varepsilon(\lambda) = \varepsilon_0(\lambda) - \int a_2(\lambda - \mu)\varepsilon(\mu)\,\mathrm{d}\mu, \tag{23}$$

where $\varepsilon_0(\lambda) = 9\pi a_1(\lambda)/2$. Using the Fourier transformation, the dressed energy of quasi-particles can be expressed as

$$\varepsilon(\lambda) = \frac{9}{8\cosh(\pi\lambda)}. \tag{24}$$

When $\lambda \to \pm\infty$, the dressed energy of quasi-particles tend to zero. Thus the elementary excitation in the present system is gapless.

The simplest elementary excitation flips one spin from the spin configuration of the ground state. In this excitation, one of the sequence quantum numbers $\{I_j\}$ is unoccupied which is equivalent to generating two holes in the λ-sea. Suppose the positions of the holes in the λ-axis are λ_1^h and λ_2^h. In this case, the distribution function $\sigma(\lambda)$ should satisfy the integral equation

$$\sigma(\lambda) + \sigma_h(\lambda) = \frac{a_1(\lambda)}{2} + \frac{a_2(\lambda)}{2} - \int a_2(\lambda - \mu)\sigma(\mu)\,\mathrm{d}\mu, \tag{25}$$

where the density of holes are quantified by

$$\sigma_h(\lambda) = \frac{1}{N}\left[\delta\left(\lambda - \lambda_1^h\right) + \delta\left(\lambda - \lambda_2^h\right)\right]. \tag{26}$$

Taking the Fourier transformation of equation (25), we obtain the density difference between the excited state and the ground state as

$$\delta\widetilde{\sigma}(\omega) = -\frac{1}{N}\frac{\mathrm{e}^{\mathrm{i}\lambda_1^h\omega} + \mathrm{e}^{\mathrm{i}\lambda_2^h\omega}}{1 + e^{-|\omega|}}. \tag{27}$$

The energy of this kind of elementary excitation can easily be derived as

$$\delta E = -\frac{9}{4}N\int a_1(\lambda)\delta\sigma(\lambda)\,\mathrm{d}\lambda = \varepsilon\left(\lambda_1^h\right) + \varepsilon\left(\lambda_2^h\right). \tag{28}$$

It is just the summation of the energy of two quasi-holes. The corresponding spin of this elementary excitation is

$$S = -N\int \delta\sigma(\lambda)\,\mathrm{d}\lambda = 1, \tag{29}$$

indicating a spin triplet excitation. Clearly, each hole carries spin-1/2, which is exactly the same as the spin-on in the spin-1/2 Heisenberg chain.

In the above investigation, we only considered the real solutions of the Bethe ansatz equation (14). In fact, the Bethe ansatz equation may have complex or string solutions with the following form [37]

$$\lambda_{j,\alpha}^{(n)} = \lambda_\alpha^{(n)} - \frac{\mathrm{i}}{2}(n + 1 - 2j) + o(e^{-\delta N}), \tag{30}$$

where $j = 1, 2, \ldots, n$, $\lambda_\alpha^{(n)}$ is a real number which represents the position of the αth n-string, δ is positive infinitesimal and the term $o(e^{-\delta N})$ means the finite size correction which can be neglected in the thermodynamic limit. The string solutions correspond to the bound states. In the thermodynamic limit, the strings are symmetrically aligned in the two sides of the real axis with the same distance. Substituting the string solutions (30) into the Bethe ansatz equation (14), we have

$$\prod_{j=1}^{n}\left(\frac{\left(\lambda_{j,\alpha}^{(n)}-\mathrm{i}/2\right)\left(\lambda_{j,\alpha}^{(n)}-\mathrm{i}\right)}{\left(\lambda_{j,\alpha}^{(n)}+\mathrm{i}/2\right)\left(\lambda_{j,\alpha}^{(n)}+\mathrm{i}\right)}\right)^{\frac{N}{2}}=\prod_{j=1}^{n}\prod_{m=1}^{\infty}\prod_{l=1,l\neq j}^{\infty}\prod_{\beta=1}^{m}\frac{\lambda_{j,\alpha}^{(n)}-\lambda_{l,\beta}^{(m)}-\mathrm{i}}{\lambda_{j,\alpha}^{(n)}-\lambda_{l,\beta}^{(m)}+\mathrm{i}}. \tag{31}$$

Taking the logarithm of equation (31), we obtain

$$\theta_n\left(\lambda_\alpha^{(n)}\right)+\theta_{2n}\left(\lambda_\alpha^{(n)}\right)=\frac{4\pi I_\alpha^{(n)}}{N}+\frac{2}{N}\sum_{m,\beta}\theta'_{mn}\left(\lambda_\alpha^{(n)}-\lambda_\beta^{(m)}\right), \tag{32}$$

where $\theta'_{mn}(\lambda)=\theta_{m+n}(\lambda)+2\theta_{m+n-2}(\lambda)+\cdots+2\theta_{|m-n|+2}(\lambda)+(1-\delta_{mn})\theta_{|m-n|}(\lambda)$. Define

$$Z_n(\lambda)=\frac{\theta_n(\lambda)}{4\pi}+\frac{\theta_{2n}(\lambda)}{4\pi}-\frac{1}{2\pi N}\sum_{j,\beta}\theta'_{m,n}\left(\lambda-\lambda_\beta^{(m)}\right).$$

In the thermodynamic limit, we denote $\mathrm{d}Z_n(\lambda)/\mathrm{d}\lambda\equiv\sigma_n(\lambda)+\sigma_n^h(\lambda)$, where $\sigma_n(\lambda)$ is the density of n-string in λ-space, and $\sigma_n^h(\lambda)$ is the density of holes. Taking the derivative of equation (32), we obtain that the densities $\sigma_n(\lambda)$ and $\sigma_n^h(\lambda)$ should satisfy

$$\sigma_n(\lambda)+\sigma_n^h(\lambda)=\frac{a_n(\lambda)}{2}+\frac{a_{2n}(\lambda)}{2}-\sum_{m=1}^{\infty}\int A_{mn}(\lambda-\mu)\sigma_m(\mu)\,\mathrm{d}\mu, \tag{33}$$

where $A_{mn}(\lambda)=a_{m+n}(\lambda)+2a_{m+n-2}(\lambda)+\cdots+2a_{|m-n|+2}(\lambda)+a_{|m-n|}(\lambda)$ and $a_0(\lambda)\equiv\delta(\lambda)$. Starting from equation (33), we can discuss the general elementary excitations and the thermodynamic properties of the system.

We consider the 2-string excitation by digging two holes in the real axis at the positions λ_1^h and λ_2^h and putting a 2-string $\lambda^s\pm\mathrm{i}/2$ in the λ-sea. Again, the density of holes $\sigma_1^h(\lambda)$ and the density of 2-string $\sigma_2(\lambda)$ are quantified by the δ-functions as

$$\sigma_1^h(\lambda)=\frac{1}{N}\left[\delta\left(\lambda-\lambda_1^h\right)+\delta\left(\lambda-\lambda_2^h\right)\right], \tag{34}$$

$$\sigma_2(\lambda)=\frac{1}{N}\delta(\lambda-\lambda^s). \tag{35}$$

The distribution functions of real λ should satisfy the integral equation

$$\begin{aligned}\sigma_1(\lambda)+\sigma_1^h(\lambda)=&\frac{a_1(\lambda)}{2}+\frac{a_2(\lambda)}{2}-\int a_2(\lambda-\mu)\sigma_1(\mu)\,\mathrm{d}\mu\\&-\int[a_1(\lambda-\mu)+a_3(\lambda-\mu)]\sigma_2(\mu)\,\mathrm{d}\mu.\end{aligned} \tag{36}$$

Substituting equations (34) and (35) into (36) and using the Fourier transformation, we obtain the change of the density of real λ respect to that of the ground state as

$$\delta\widetilde{\sigma}_1(\omega)=-\frac{\mathrm{e}^{\mathrm{i}\lambda_1^h\omega}+\mathrm{e}^{\mathrm{i}\lambda_2^h\omega}}{N(1+e^{-|\omega|})}-\frac{e^{-\frac{1}{2}|\omega|}+e^{-\frac{3}{2}|\omega|}}{N(1+e^{-|\omega|})}\mathrm{e}^{\mathrm{i}\lambda_s w}. \tag{37}$$

The corresponding excitation energy therefore reads

$$\begin{aligned}\delta E&=-\frac{9\pi N}{2}\left[\int a_1(\lambda)\delta\sigma_1(\lambda)\,\mathrm{d}\lambda+\frac{1}{N}a_2(\lambda_s)\right]\\&=\varepsilon\left(\lambda_1^h\right)+\varepsilon\left(\lambda_2^h\right).\end{aligned} \tag{38}$$

We find that the excitation energy is the same in form as that of the spin triplet excitation. The excitation energy only depends on the positions of the holes. The contribution of the 2-string is completely canceled by the rearrangement of the λ-sea after digging holes. The spin carried by this excitation is

$$S = \frac{3}{4}N - \frac{1}{4}N - N\int \sigma_1(\lambda)\,\mathrm{d}\lambda - 2\int \sigma_2(\lambda)\,\mathrm{d}\lambda = 0,$$

which means it is a spin singlet excitation or a spinon–antispinon pair excitation.

5. Conclusion

In summary, we construct an integrable ferrimagnetic spin chain model. The exact energy spectrum is obtained by using the algebraic Bethe ansatz method. Unlike the other integrable mixed spin chain model, our model has a ferrimagnetic ground state with a finite residual magnetization. It is also found that the low-lying elementary excitations can be well described by the usual spinons which carry a fractional spin-1/2. Using the standard thermodynamic Bethe ansatz method [37–41], the finite temperature properties of the system can be obtained directly.

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